PROPERTIES OF MEMBERSHIP FUNCTIONS, FUZZIFICATION AND DEFUZZIFICATION

FEATURES OF THE MEMBERSHIP FUNCTION

Since all information contained in a fuzzy set is described by its membership function, it is useful to develop a lexicon of terms to describe various special features of this function. For purposes of simplicity, the functions shown in the following figures will all be continuous, but the terms apply equally for both discrete and continuous fuzzy sets. Figure 5.1 assists in this description.

The ***core*** of a membership function for some fuzzy set A~ is defined as that region of the universe that is characterized by complete and full membership in the set A~. That is, the core comprises those elements *x* of the universe such that $\mu_{A\sim}(x) = 1$.

The ***support*** of a membership function for some fuzzy set A~ is defined as that region of the universe that is characterized by nonzero membership in the set A~. That is, the support comprises those elements *x* of the universe such that $\mu_{A\sim}(x) > 0$.

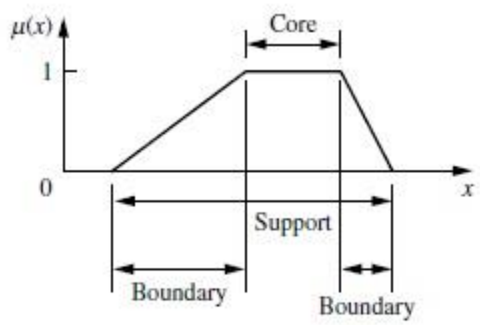


FIGURE 5.1: Core, support, and boundaries of a fuzzy set.

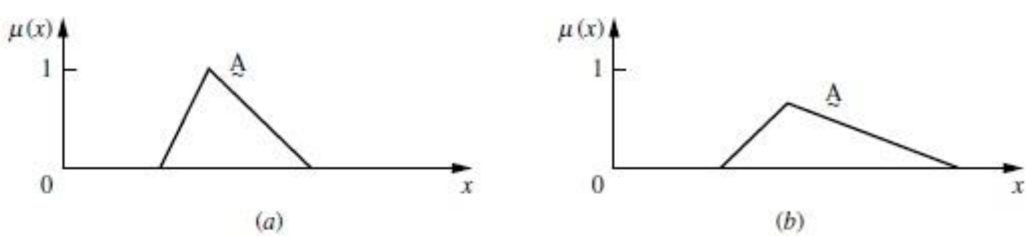


FIGURE 5.2: Fuzzy sets that are normal (*a*) and subnormal (*b*).

The ***boundaries*** of a membership function for some fuzzy set A~ are defined as that region of the universe containing elements that have a nonzero membership but not complete membership. That is, the boundaries comprise those elements *x* of the universe such that $0 < \mu_{A\sim}(x) < 1$. These elements of the universe are those with some *degree* of fuzziness, or only partial membership in the fuzzy set A~. Figure 5.1 illustrates the regions in the universe comprising the core, support, and boundaries of a typical fuzzy set.

A *normal* fuzzy set is one whose membership function has at least one element *x* in the universe whose membership value is unity. For fuzzy sets where one and only one element has a membership equal to one, this element is typically referred to as the *prototype* of the set, or the prototypical element. Figure 5.2 illustrates typical normal and subnormal fuzzy sets.
A *convex* fuzzy set is described by a membership function whose membership values are strictly monotonically increasing, or whose membership values are strictly monotonically decreasing, or

whose membership values are strictly monotonically increasing then strictly monotonically decreasing with increasing values for elements in the universe.

FUZZIFICATION

Fuzzification is the process of making a crisp quantity fuzzy. We do this by simply recognizing that many of the quantities that we consider to be crisp and deterministic are actually not deterministic at all: They carry considerable uncertainty. If the form of uncertainty happens to arise because of imprecision, ambiguity, or vagueness, then the variable is probably fuzzy and can be represented by a membership function.

In the real world, hardware such as a digital voltmeter generates crisp data, but these data are subject to experimental error. The information shown in Fig.5.3 shows one possible range of errors for a typical voltage reading and the associated membership function that might represent such imprecision.

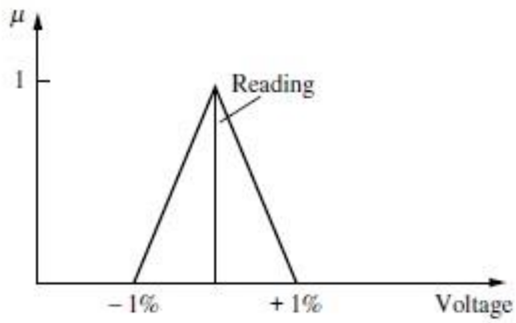


FIGURE 5.3: Membership function representing imprecision in ''crisp voltage reading.''

DEFUZZIFICATION TO CRISP SETS

We begin by considering a fuzzy setA∼, then define a lambda-cut set, A_λ, where $0 \le \lambda \le 1$.

The set A_λ is a crisp set called the lambda (λ)-cut (or alpha-cut) set of the fuzzy set A∼, where $A_\lambda = \{x | \mu_{A\sim}(x) \ge \lambda\}$. Note that the λ-cut set A_λ does not have a tilde underscore; it is a crisp set derived from its parent fuzzy set, A∼. Any particular fuzzy set A∼ can be transformed into an infinite number of λ-cut sets, because there are an infinite number of values λ on the interval [0, 1].

Any element $x \in A_\lambda$ belongs to A∼ with a grade of membership that is greater than or equal to the value λ. The following example illustrates this idea.

Example 5.1. Let us consider the discrete fuzzy set, using Zadeh's notation, defined on universe X = $\{a, b, c, d, e, f\}$,

$$\underset{\sim}{A} = \left\{ \frac{1}{a} + \frac{0.9}{b} + \frac{0.6}{c} + \frac{0.3}{d} + \frac{0.01}{e} + \frac{0}{f} \right\}$$

This fuzzy set is shown schematically in Fig. 4.8. We can reduce this fuzzy set into several λ-cut sets, all of which are crisp. For example, we can define λ-cut sets for the values of $\lambda = 1$, 0.9, 0.6, 0.3, 0^+, and 0.

$$A_1 = \{a\}, \quad A_{0.9} = \{a, b\}$$
$$A_{0.6} = \{a, b, c\}, \quad A_{0.3} = \{a, b, c, d\}$$
$$A_{0^+} = \{a, b, c, d, e\}, \quad A_0 = X$$

Example 5.2. Supposewe take the fuzzy relation from the biotechnology example, and perform λ-cut operations for the values of λ = 1, 0.9, 0. These crisp relations are given below:

$$\lambda = 1,\ R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\lambda = 0.9,\ R_{0.9} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$\lambda = 0$, $R_0 = \mathbf{E}\sim$ (whole relation)

DEFUZZIFICATION TO SCALARS

As mentioned in the introduction, there may be situations where the output of a fuzzy process needs to be a single scalar quantity as opposed to a fuzzy set. Defuzzification is the conversion of a fuzzy quantity to a precise quantity, just as fuzzification is the conversion of a precise quantity to a fuzzy quantity. The output of a fuzzy process can be the logical union of two or more fuzzy membership functions defined on the universe of discourse of the output variable. For example, suppose a fuzzy output is comprised of two parts: the first part,$C_1\sim$, a trapezoidal shape, shown in Fig. 5.4*a*, and the second part,$C_2\sim$, a triangular membership shape, shown in Fig. 5.4*b*. The union of these two membership functions, i.e., $C\sim = C_1\sim \cup C_2\sim$, involves the max operator, which graphically is the outer envelope of the two shapes shown in Figs. 4.11*a* and *b*; the resulting shape is shown in Fig. 5.4*c*. Ofcourse, a general fuzzy output process can involve many output parts (more than two), and the membership function representing each part of the output can have shapes other than triangles and trapezoids. Further, as Fig. 5.4*a* shows, the membership functions may not always be normal.

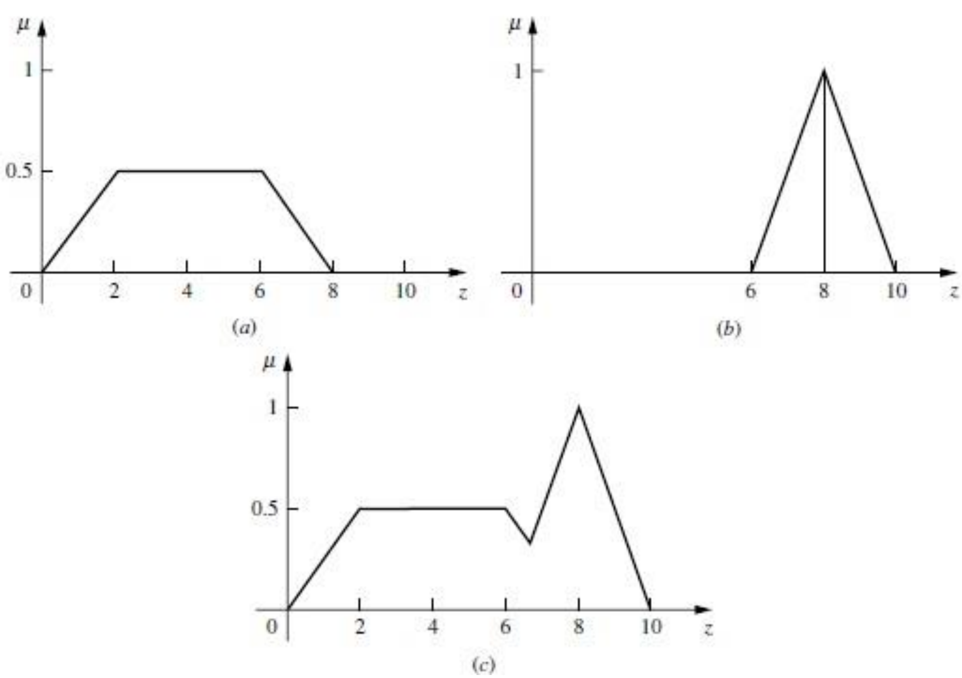


Figure 5.4

Seven (7) methods of defuzzification into scalar are discussed below:

***Weighted average method*:** The weighted average method is the most frequently used in fuzzy applications since it is one of the more computationally efficient methods.
Unfortunately it is *usually* restricted to symmetrical output membership functions. It is given by the algebraic expression

$$z^* = \frac{\sum \mu_{\underset{\sim}{C}}(\bar{z}) \cdot \bar{z}}{\sum \mu_{\underset{\sim}{C}}(\bar{z})}$$

Where denotes the algebraic sum and where *z* is the centroid of each symmetric membership function. This method is shown in Fig. 5.5. The weighted average method is formed by weighting each membership function in the output by its respective maximum membership value. As an example, the two functions shown in Fig. 5.5 would result in the following general form for the defuzzified value

$$z^* = \frac{a(0.5) + b(0.9)}{0.5 + 0.9}$$

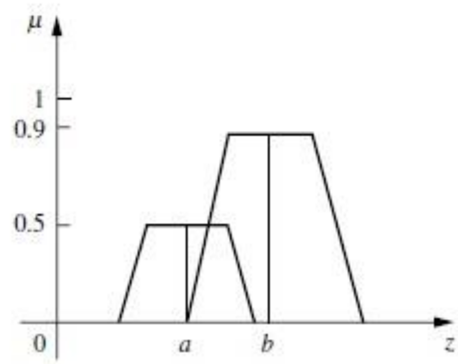


Figure 5.5

***Centroid method*:** This procedure (also called center of area, center of gravity) is the
most prevalent and physically appealing of all the defuzzification methods [Sugeno, 1985; Lee, 1990]; it is given by the algebraic expression

$$z^* = \frac{\int \mu_{\underset{\sim}{C}}(z) \cdot z\,dz}{\int \mu_{\underset{\sim}{C}}(z)\,dz}$$

Max membership principle: Also known as the height method, this scheme is limited to
peaked output functions. This method is given by the algebraic expression

$$\mu_{\underset{\sim}{C}}(z^*) \geq \mu_{\underset{\sim}{C}}(z) \qquad \text{for all } z \in Z$$

Mean max membership: This method (also called middle-of-maxima) is closely related
to the first method, except that the locations of the maximum membership can be
nonunique (i.e., the maximum membership can be a plateau rather than a single point).
This method is given by the expression

$Z^* = \frac{a+b}{2}$

***Center of sums*:** This is faster than many defuzzification methods that are presently in use, and the method is not restricted to symmetric membership functions. This process involves the algebraic sum of individual output fuzzy sets, say$C_1\sim$ and $C_2\sim$, instead of their union. Two drawbacks to this method are that the intersecting areas are added twice, and the method also involves finding the centroids of the individual membership functions. The defuzzified value z^* is given by the following equation:

$$z^* = \frac{\int_Z \bar{z} \sum_{k=1}^{n} \mu_{C_k}(z)\, dz}{\int_z \sum_{k=1}^{n} \mu_{C_k}(z)\, dz}$$

where the symbol z is the distance to the centroid of each of the respective membership functions. This method is similar to the weighted average method, except in the center of sums method the weights are the areas of the respective membership functions whereas in the weighted average method the weights are individual membership values.

Center of largest area: If the output fuzzy set has at least two convex subregions, then the center of gravity (i.e., z^* is calculated using the centroid method of the convex fuzzy subregion with the largest area is used to obtain the defuzzified value z^* of the output.

$$z^* = \frac{\int \mu_{C_m}(z) z\, dz}{\int \mu_{C_m}(z)\, dz}$$

whereC_m~ is the convex subregion that has the largest area making up C_k~. This condition applies in the case when the overall output C_k~ is nonconvex; and in the case when C_k~ is convex, z^* is the same quantity as determined by the centroid method or the center of largest area method (because then there is only one convex region).

First (or last) of maxima: This method uses the overall output or union of all individual output fuzzy sets C_k~ to determine the smallest value of the domain with maximized membership degree in C_k~.

Example 5.3.

A railroad company intends to lay a new rail line in a particular part of a county.
The whole area through which the new line is passing must be purchased for right-of-way considerations. It is surveyed in three stretches, and the data are collected for analysis. The surveyed data for the road are given by the sets, B_1~, B_2~, and B_3~, where the sets are defined on the universe of right-of-way widths, in meters. For the railroad to purchase the land, it must have an assessment of the amount of land to be bought. The three surveys on right-of-way width are ambiguous, however, because some of the land along the proposed railway route is already public domain and will not need to be purchased. Additionally, the original surveys are so old that some ambiguity exists on boundaries and public right-of-way for old utility lines and old roads. The three fuzzy sets, B_1~, B_2~, and B_3~, shown in Figs. 5.6, 5.7, and 5.8, respectively, represent the uncertainty in each survey as to the membership of right-of-way width, in meters, in privately owned land. We now want to aggregate these three survey results to find the single most nearly representative right-of-way width (z) to allow the railroad to make its initial estimate of the right-of-way purchasing cost using Weighted Average method and Centroid method.

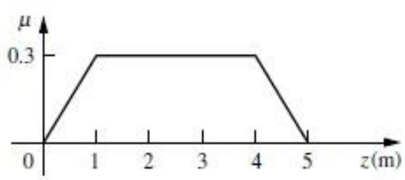


FIGURE 5.6: Fuzzy set B_1~: public right-of-way width (z) for survey 1.

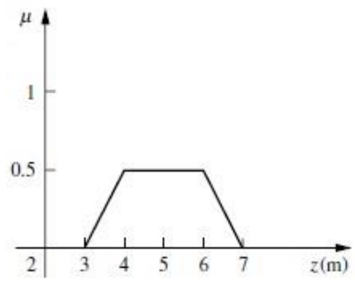


FIGURE 5.7: Fuzzy set B_2~: public right-of-way width (z) for survey 2.

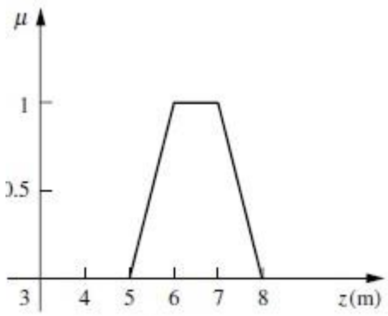
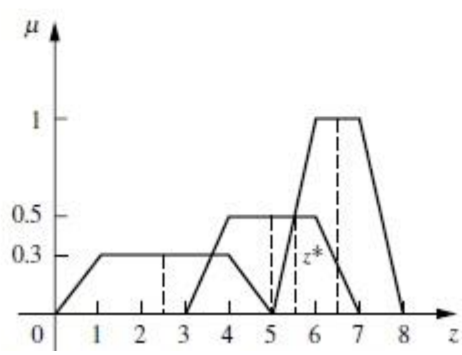


FIGURE 5.8: Fuzzy set B_3~: public right-of-way width (z) for survey 3.

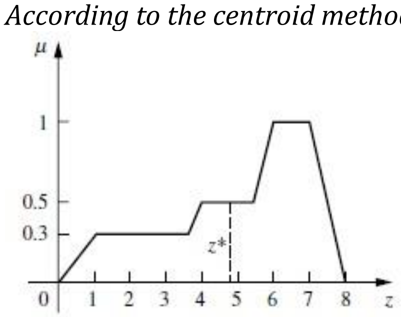
Solution

According to the weighted average method

$$z^* = \frac{(0.3\times2.5)+(0.5\times5)+(1\times6.5)}{0.3+0.5+1} = 5.41\text{m}$$



According to the centroid method



$$z^* = \frac{\int \mu_{\underset{\sim}{C}}(z) \cdot z \, dz}{\int \mu_{\underset{\sim}{C}}(z) \, dz}$$

Z* = $\left[\int_0^1 (0.3z)zdz + \int_1^{3.6}(0.3z)dz + \int_{3.6}^4 \left(\frac{z-3.6}{2}\right) zdz + \int_4^{5.5}(0.5z)dz + \int_{5.5}^6 \left(\frac{z-5.5}{2}\right) zdz + \int_6^7 zdz + \int_7^8 \left(\frac{7-z}{2}\right) zdz \right] / \left[\int_0^1 (0.3z)dz + \int_1^{3.6}(0.3)dz + \int_{3.6}^4 \left(\frac{z-3.6}{2}\right) dz + \int_4^{5.5}(0.5)dz + \int_{5.5}^6 \left(\frac{z-5.5}{2}\right) dz + \int_6^7 1dz + \int_7^8 \left(\frac{7-z}{2}\right) dz \right]$

= 10.995/2.54 = 4.33m

Z* = 4.33m